1. A chicken lays n eggs. Each egg independently does or doesn’t hatch, with probability p of hatching. For each egg that hatches, the chick does or doesn’t survive (independently of the other eggs), with probability s of survival. Let N ⇠ Bin(n, p) be the number of eggs which hatch, X be the number of chicks which survive, and Y be the number of chicks which hatch but don’t survive (so X + Y = N). Find the marginal PMF of X, and the joint PMF of X and Y . Are they independent?

Answer:

To find the marginal PMF of X, we need to sum the joint PMF of X and Y over all possible values of Y. Since X + Y = N, we can write the joint PMF as:

P(X=x, Y=y) = P(X=x, N=x+y) = binom(x+y, x) \* p^(x+y) \* (1-p)^(n-x-y) \* binom(n-x-y, y) \* s^x \* (1-s)^y

where binom(a, b) denotes the binomial coefficient "a choose b".

To find the marginal PMF of X, we need to sum this joint PMF over all possible values of Y:

P(X=x) = sum(P(X=x, Y=y), y=0 to n-x) = sum(binom(x+y, x) \* p^(x+y) \* (1-p)^(n-x-y) \* binom(n-x-y, y) \* s^x \* (1-s)^y, y=0 to n-x)

The joint PMF of X and Y is not independent because P(X=x, Y=y) ≠ P(X=x)P(Y=y) for all x and y. To see this, note that P(X=x) is a function of p and s, while P(Y=y) is only a function of p. Therefore, the joint PMF of X and Y depends on both p and s, and cannot be factored into a product of two functions of p and s, one depending only on X and the other depending only on Y.

In other words, the survival of a chick depends on whether its egg hatches or not, so the number of chicks which survive and the number of chicks which hatch but don't survive are not independent. If we know the number of chicks which survive, we can infer information about the number of chicks which hatch but don't survive, and vice versa.